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# Products of projections in von Neumann algebras☆

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**Abstract**

We describe the elements of von Neumann algebras which can be represented as products of orthogonal projections and idempotents, and estimate the minimal number of terms in the product.


*Keywords:* von Neumann algebras; Projections; Idempotents

## 1. Introduction and the main results

We investigate whether an element $a$ in a von Neumann algebra $\mathscr{N}$ can be represented as a product of projections in $\mathscr{N}$, and if yes, what is the minimal number of projections required (this number is denoted by $\mathbf{M}(a)$).

Throughout this paper, all von Neumann algebras and $C^*$-algebras are assumed to be acting on a fixed separable Hilbert space. An element $a \in \mathscr{N}$ is called *an idempotent* if $a = a^2$. A self-adjoint idempotent is called *a projection*. For a von Neumann algebra $\mathscr{N}$, we denote the set of its projections by $\mathscr{P}(\mathscr{N})$. If $E$ is a subspace of a Hilbert space $H$, $\mathbf{pr}(E)$ denotes the projection onto the closure of $E$. $E$ is said to be *affiliated* with a von Neumann algebra $\mathscr{N}$ if $\mathbf{pr}(E) \in \mathscr{N} \hookrightarrow B(H)$ (equivalently, by [10], $E$ is the range of an idempotent in $\mathscr{N}$).

Throughout, **ran** and **ker** denote the range and kernel of an operator, respectively. The usual Murray–von Neumann relations on $\mathscr{P}(\mathscr{N})$ are denoted by $\prec$, $\succ$, and $\sim$. For $p, q \in \mathscr{P}(\mathscr{N})$, we say that $p$ *$n$-majorizes* $q$ ($p \gg_n q$) if there exist $n$ mutually orthogonal projections $q_1, \ldots, q_n \in \mathscr{N}$ such that $q = q_1 + \cdots + q_n$, and $q_j$ is equivalent to a subprojection of $p$ for $1 \leqslant j \leqslant n$. We

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say that a projection $p \in \mathcal{N}$ is *n-majorant* if $p \gg_n \mathbf{1}_{\mathcal{N}} - p = p^\perp$. Similar notation is used for subspaces affiliated with $\mathcal{N}$, which are identified with the projections onto them. For instance, we say that subspaces $E$ and $F$ are equivalent, and write $E \sim F$, if $\mathbf{pr}(E) \sim \mathbf{pr}(F)$.

**Theorem 1.1.** *Suppose $\mathcal{N}$ is a von Neumann algebra, acting on a separable Hilbert space $H$.*

(1) *Suppose $p \in \mathcal{P}(\mathcal{N})$, $a \in \mathcal{N}$ satisfy $a = p^\perp + pap$, $\|pap\| < 1$, $p \ll_n \mathbf{ker}\, a$, and $(\mathbf{ran}\, a)^\perp \sim \mathbf{ker}\, a$. Then $a$ can be represented as a product of at most $\gamma n/(1 - \|pap\|)$ projections ($\gamma$ is an absolute constant).*
(2) *Suppose $a = p_n \dots p_1$, with $p_1, \dots, p_n \in \mathcal{P}(\mathcal{N})$. Then $\mathbf{ker}\, a \sim (\mathbf{ran}\, a)^\perp$, and there exists $p \in \mathcal{P}(\mathcal{N})$ such that $a = p^\perp + pap$, with $\|a\xi\| < \|\xi\|$ for any $\xi \in \mathbf{ran}\, p\backslash\{0\}$, and $p - \mathbf{pr}(\mathbf{ran}\,(pap)) \ll_{n-1} \mathbf{pr}(\mathbf{ker}\, a)$.*

The estimate on the minimal number of projections in part (1) is optimal:

**Proposition 1.2.** (1) *Suppose $p$ is a projection in a von Neumann algebra $\mathcal{N}$, such that $\mathbf{ker}\, p$ is $k$-majorant, but not $(k-1)$-majorant. Then $\mathbf{M}(-\lambda p) \geqslant \max\{k, (1+\lambda)/(1-\lambda)\}$.*

(2) *Moreover, suppose $\tau$ is a faithful normal finite trace on a von Neumann algebra $\mathcal{N}$, $p \in \mathcal{P}(\mathcal{N})$, and $\lambda \in (0, 1)$. Then*
$$\mathbf{M}(-\lambda p) \geqslant \frac{\tau(p)}{1-\tau(p)} \frac{1+\lambda}{1-\lambda}.$$

More can be said about self-adjoint elements.

**Theorem 1.3.** *Suppose $\mathcal{N}$ is a von Neumann factor, and $a \in \mathcal{N}$ is self-adjoint. Then $a$ is a product of projections if and only if $\sigma(a) \subset (-1, 1]$, and $\mathbf{ker}\, a$ is $n$-majorant, for some $n$. If $\mathbf{ker}\, a$ is $n$-majorant, and $\sigma(a) \subset [-\lambda, 1]$ for some $\lambda \in [0, 1)$, then $\mathbf{M}(a) \leqslant \gamma' n/(1-\lambda)$, where $\gamma'$ is a constant.*

Furthermore, we can describe precisely which positive operators can be represented as symmetric products of projections.

**Theorem 1.4.** *Suppose $0 \leqslant a \leqslant \mathbf{1}$ is an element of a von Neumann algebra $\mathcal{N}$, and $n$ is a natural number. Then the following two statements are equivalent:*

(1) $\overline{\mathbf{ran}\, a} \ominus \mathbf{ker}\,(\mathbf{1} - a) \ll_n \mathbf{ker}\, a$.
(2) $a = p_1 \dots p_n p_{n+1} p_n \dots p_1$ *for some* $p_1, \dots, p_{n+1} \in \mathcal{P}(\mathcal{N})$.

A similar result was obtained in [1], in relation to almost sharp quantum effects.

Observe that our estimate for the minimal number of projections in a "symmetric" product representing $a$ is sharp: if $\overline{\mathbf{ran}\, a} \ominus \mathbf{ker}\,(\mathbf{1} - a) \ll_{n-1} \mathbf{ker}\, a$ is not true, then we cannot write $a = q_1 \dots q_{n-1} q_n q_{n-1} \dots q_1$. Indeed, suppose, for the sake of contradiction, that the representation as above exists. By the observations in the beginning of Section 2.1, it suffices to consider the case of $\mathbf{ker}\,(\mathbf{1} - a) = 0$. Let $u = q_n q_{n-1} \dots q_1$, and observe that $a = u^* u$. By Lemma 2.15, $\overline{\mathbf{ran}\, u} \ll_{n-1} \mathbf{ker}\, u$. However, $\overline{\mathbf{ran}\, u} \sim \overline{\mathbf{ran}\, a}$, yielding a contradiction.

We also describe the closure of the set of products of projections in the strong or weak operator topologies.

**Theorem 1.5.** *Denote by* $\boldsymbol{\tau}$ *the weak operator topology, the strong operator topology, or the weak* topology on a separably acting von Neumann algebra* $\mathscr{N}$. *Then the* $\boldsymbol{\tau}$*-closure of all products of projections in* $\mathscr{N}$ *coincides with the unit ball of* $\mathscr{N}$ *if and only if* $\mathscr{N}$ *has no* $M$*-summands of type* $I_n$ $(n \in \mathbb{N})$.

Finally, we deal with the products of idempotents (not necessarily self-adjoint) in von Neumann algebras. We say that subspaces $E$ and $F$ of a Hilbert space $H$ are *at positive angles* with each other if $\|\mathbf{pr}(E)\mathbf{pr}(F)\| < 1$, or equivalently, $\mathbf{pr}(E)^{\perp}|_F$ is an isomorphism.

**Theorem 1.6.** *Consider an element* $u$ *of a von Neumann algebra* $\mathscr{N}$, *acting on a separable Hilbert space* $H$.

(1) *If* $u$ *is a product of* $n$ *idempotents from* $\mathscr{N}$, *then* $\mathbf{ker}\, u \sim (\mathbf{ran}\, u)^{\perp}$, *and there exists a subspace* $E$, *affiliated with* $\mathscr{N}$, *such that* (i)$u|_E = I_E$, (ii) $E$ *and* $\overline{u(F)}$ *are at positive angles, where* $F = H \ominus (E + \mathbf{ker}\, u)$, *and* (iii) $F \ll_{n-1} \mathbf{ker}\, u$. *If* $\mathbf{ker}\, u$ *is trivial, then* $E = H$.
(2) *Conversely, suppose for for a given* $u$ *there exist* $E$, $F$, *and* $n$ *as above. Then* $u$ *is a product of at most* $\gamma''n$ *idempotents, where* $\gamma''$ *is a constant.*

Note that, in part (1), $E$ is at positive angles with $\mathbf{ker}\, u$.

The minimal number of idempotents needed to represent an element of $\mathscr{N}$ is not known to us. It is tempting to conjecture that this number coincides with $n$ from part (1) of the theorem above. Indeed, this is true for finite type I factors [2]. However, the "conjecture" fails for infinite factors, see Propositions 3.6 and 3.4, as well as [5].

For the particular case of the von Neumann algebras $B(H)$, some descriptions of products of projections were obtained by the author in [13]. That paper also contains references to other related articles. An overview of products of other types of operators can be found in [16]. Linear combinations of products of projections of fixed length were described in [3].

## 2. Proofs of the main results: products of projections

### 2.1. Preliminary notes

(i) Suppose $p_1, \ldots, p_n$ are projections in a von Neumann algebra $\mathscr{N}$ (acting on a Hilbert space $H$), and $p = p_1 \wedge p_2 \wedge \cdots \wedge p_n$. Then, for each $j$, $p'_j := p_j p^{\perp} = p^{\perp} p_j = p_j - p$ is a projection. Moreover, $p_n \ldots p_1 = p + p'_n \ldots p'_1$. The domain and range of $p'_n \ldots p'_1$ are orthogonal to $\mathbf{ran}\, p$. Finally, $\mathbf{ran}\, p = \{\xi \in H \mid \|p_n \ldots p_1 \xi\| = \|\xi\|\}$.

(ii) Suppose $\mathscr{I}$ is an index set, $(\mathscr{N}_\alpha)_{\alpha \in \mathscr{I}}$ are von Neumann algebras, and $u_\alpha \in \mathscr{N}_\alpha$ for each $\alpha$. Let $\mathscr{N} = \oplus_{\alpha \in \mathscr{I}} \mathscr{N}_\alpha$, and $u = \sum_{\alpha \in \mathscr{I}} u_\alpha$. Then $u$ is a product of $n$ projections iff $u_\alpha$ is a product of $n$ projections for each $\alpha$.

(iii) Suppose $u \in \mathscr{N}$, and $p$ is the projection onto $\mathbf{ker}\,(u - \mathbf{1}_{\mathscr{N}}) = \{\xi \in H \mid u\xi = \xi\}$. Let $\mathscr{N}_1 = p^{\perp} \mathscr{N} p^{\perp}$, and $u_1 = p^{\perp} u p^{\perp}$. Then the following are equivalent:

(1) $u$ is a product of $n$ projections in $\mathscr{N}$.
(2) $u_1$ is a product of $n$ projections in $\mathscr{N}_1$.

Indeed, (2) $\Rightarrow$ (1) follows directly from Observation (ii). To see the converse, suppose $u = p_n \dots p_1$. Then, by (i), $u_1 = (p_n - p) \dots (p_1 - p)$.

(iv) Because of (i) and (iii), we are especially interested in those $u \in \mathcal{N} \hookrightarrow B(H)$ for which $\|u\xi\| < \|\xi\|$ for any $\xi \in H\backslash\{0\}$. However, characterization of such elements is difficult, see [13]. To make the problem more manageable, we concentrate on the case $\|u\| < 1$.

**Lemma 2.1.** *Suppose* $u = q_n \dots q_1$, *for* $q_1, \dots, q_n \in \mathscr{P}(\mathcal{N})$. *Then there exist* $p_1, \dots, p_n \in \mathscr{P}(\mathcal{N})$ *such that* $u = p_n \dots p_1$, $\mathbf{ker}\, u = \mathbf{ker}\, p_1$, $\mathbf{ran}\, p_j = \overline{\mathbf{ran}\, (p_j p_{j-1} \dots p_1)}$, *and, for* $2 \leqslant j \leqslant n$, $p_j$ *is surjective on* $\mathbf{ran}\, (p_{j-1} \dots p_1)$.

**Proof.** Let $p_1 = q_1 \wedge \mathbf{pr}(\mathbf{ker}\, u)^\perp$. Then $u = q_n \dots q_2 p_1$, and $\mathbf{ker}\, u = \mathbf{ker}\, p_1$. Therefore, $\mathbf{ker}\, q_k \cap \mathbf{ran}\, (q_{k-1} \dots q_2 p_1) = \{0\}$ for $2 \leqslant k \leqslant n$.

Suppose $p_1, \dots, p_k$ have already been defined ($1 \leqslant k < n$) in such a way that $\mathbf{ran}\, p_j = \overline{\mathbf{ran}\, (p_j p_{j-1} \dots p_1)}$ for $j \leqslant k$, and $u = q_n \dots q_{k+1} p_k \dots p_1$. Let $p_{k+1} = \mathbf{pr}(\mathbf{ran}(q_{k+1} p_k \dots p_1))$. Clearly, this projection has the desired properties: $u = q_n \dots q_{k+2} p_{k+1} \dots p_1$, and $p_{k+1}$ is surjective on $\mathbf{ran}(p_k \dots p_1)$ (otherwise, $\mathbf{ker}\, u$ strictly contains $\mathbf{ker}\, p_1$, which contradicts our definition of $p_1$). $\square$

Henceforth, we will always assume that any product $u = p_n \dots p_2 p_1$, the projections $p_1, \dots, p_n$ satisfy the conditions of the lemma above.

Below we collect some facts about the relation $\ll_n$. Note first that, if $p, q \in \mathscr{P}(\mathcal{N})$ are such that $p \ll_n q$, and $e \in \mathscr{P}(\mathcal{N})$ is central, then $pe \ll_n qe$. Conversely, suppose $(e_\alpha)_{\alpha \in \mathscr{I}}$ is a family of central projections in $\mathcal{N}$, whose sum equals $\mathbf{1}$. If $p, q \in \mathscr{P}(\mathcal{N})$ satisfy $pe_\alpha \ll_n qe_\alpha$ for each $\alpha$, then $p \ll_n q$.

**Lemma 2.2.** *Suppose* $p, q, r \in \mathscr{P}(\mathcal{N})$, *and* $r \leqslant q \ll_n p$. *Then* $r \ll_n p$.

**Proof.** Write $q = q_1 + \cdots + q_n$ (a sum of $n$ mutually orthogonal projections, dominated by $p$). Our goal is to write $r$ as a sum of $n$ mutually orthogonal projections $r_1, \dots, r_n$, dominated by $p$. Let $r_1 = \mathbf{pr}(\mathbf{ran}\, (rq_1))$. Clearly, $r_1 \prec q_1 \prec p$.

Now suppose $r_1, \dots, r_k$ have been defined in such a way that, for $1 \leqslant i \leqslant k$, (1) $r_i \prec q_i$, and (2) $r_1 + \cdots + r_i = \mathbf{pr}(\mathbf{ran}\, (r(q_1 + \cdots + q_i)))$. Set $r_{k+1} = \mathbf{pr}(\mathbf{ran}\, (r(q_1 + \ldots + q_{k+1}))) - (r_1 + \cdots + r_k)$. Then $r_{k+1} r q_i = 0$ for $1 \leqslant i \leqslant k$, hence

$$\begin{aligned} r_{k+1} &= \mathbf{pr}(\mathbf{ran}\, (r_{k+1} r (q_1 + \cdots + q_{k+1}))), \\ &= \mathbf{pr}(\mathbf{ran}\, (r_{k+1} r q_{k+1})) \prec q_{k+1} \prec p, \end{aligned}$$

as desired. $\square$

**Lemma 2.3.** *Suppose* $p \in \mathscr{P}(\mathcal{N})$ *is n-majorant*, $q \in \mathscr{P}(\mathcal{N})$, *and* $\|p - q\| < 1$. *Then $q$ is n-majorant.*

**Proof.** We have: $\|p - q\| = \|p^\perp - q^\perp\| < 1$. Then (see e.g. [17]) there exists a unitary $u \in \mathcal{N}$ s.t. $q = u^* p u$ and $q^\perp = u^* p^\perp u$. Now suppose $p^\perp = p_1 + \cdots + p_n$, where $p_1, \dots, p_n$ are equivalent to subprojections of $p$. Taking $q_k = u^* p_k u$ ($1 \leqslant k \leqslant n$), we are done. $\square$

Next we describe $n$-majoration in specific types of von Neumann algebras. By Section 1.22 of [14], for any $k \in \mathbb{N} \cup \{0\}$, any separably acting von Neumann algebra of type $I_k$ can be represented as $L_\infty(\Omega, \mu, B(\ell_2^k))$, where $\mu$ is a $\sigma$-finite measure on $\Omega$ (we take $\ell_2^\infty$ to mean $\ell_2$).

**Lemma 2.4.** *Suppose $\mu$ is a $\sigma$-finite measure on $\Omega$, and $n \in \mathbb{N}$.*

(a) *Suppose $\mathscr{N} = L_\infty(\Omega, \mu, M_k)$ $(k \in \mathbb{N})$. Then $p \in \mathscr{P}(\mathscr{N})$ is $n$-majorant iff* $\operatorname{rank} p(t) \geqslant m = \lceil k/(n+1) \rceil$ *for almost every $t \in \Omega$.*
(b) *Suppose $\mathscr{N} = L_\infty(\Omega, \mu, B(\ell_2))$. Then $p \in \mathscr{P}(\mathscr{N})$ is $n$-majorant iff $p(t)$ has infinite rank for almost every $t \in \Omega$. In this case, $p \succ p^\perp$.*
(c) *Suppose $\mathscr{N}$ is a von Neumann algebra of type $I_\infty$, $II_\infty$, or $III$. Then $p \in \mathscr{P}(\mathscr{N})$ is $n$-majorant iff $p \succ p^\perp$.*
(d) *If $\mathscr{N}$ is a type* III *von Neumann algebra, then $p \in \mathscr{P}(\mathscr{N})$ is $n$-majorant iff $p \neq 0$.*

The following technical result must be known to specialists, but we haven't been able to find a reference to it in the literature.

**Lemma 2.5.** *Suppose $\mathscr{N}$ is a von Neumann algebra, $\mu$ is a $\sigma$-finite measure on $\Omega$, and $p \in \mathscr{P}(L_\infty(\Omega, \mu, \mathscr{N}))$. Then for every $\varepsilon > 0$ there exist disjoint measurable subsets $S_i \subset \Omega$ $(i \in \mathbb{N})$, and projections $p_i \in \mathscr{P}(\mathscr{N})$, such that $\|p - \sum_i \chi_{S_i} \otimes p_i\| < \varepsilon$.*

**Proof.** Suppose $\varepsilon \in (0, 1/2)$. It is known (see e.g. Section II.1 of [6]) that separably valued functions are dense in $L_\infty(\Omega, \mu, \mathscr{N})$. Thus, there exist $(S_i)$ as in the statement of the lemma, and $a_i \in \mathscr{N}$ s.t. $\|p - a\| < \varepsilon/2$, where $a = \sum_i \chi_{S_i} \otimes a_i$. By passing from $a_i$ to $(a_i + a_i^*)/2$, we can assume that all the $a_i$'s are self-adjoint. Note that $\|p - \lambda\| = \max\{|\lambda|, |1-\lambda|\}$, hence $\sigma(a) \subset [-\varepsilon/2, \varepsilon/2] \cup [1 - \varepsilon/2, 1 + \varepsilon/2]$. Set

$$f(s) = \begin{cases} 0, & s \leqslant \varepsilon/2, \\ (s - \varepsilon/2)/(1-\varepsilon), & \varepsilon/2 \leqslant s \leqslant 1 - \varepsilon/2, \\ 1, & s \geqslant 1 - \varepsilon/2, \end{cases}$$

and let $q = f(a) = \sum_i \chi_{S_i} f(a_i)$. Then $q$ is a projection, and $\|q - a\| = \max_{s \in \sigma(a)} |f(s) - s| \leqslant \varepsilon$. $\square$

**Proof of Lemma 2.4.** (a) If $p$ is $n$-majorant, then, for almost every $t \in \Omega$, $p(t)$ is $n$-majorant in $M_k$. Therefore, $n \operatorname{rank} p(t) \geqslant \operatorname{rank} p(t)^\perp = k - \operatorname{rank} p(t)$, which yields $\operatorname{rank} p(t) \geqslant m$.

Suppose $\operatorname{rank} p(t) \geqslant m$ for almost every $t$. By Lemmas 2.5 and 2.3, we can assume that $p = \sum_{i \in \mathbb{N}} \chi_{S_i} \otimes p^{(i)}$, where $p^{(i)} \in \mathscr{P}(M_k)$ for each $i$, and $\cup_i S_i = \Omega$. Then we can write $\mathbf{1}_{M_k} - p^{(i)} = \sum_{j=1}^n p_j^{(i)}$, with $\operatorname{rank} p_j^{(i)} \leqslant m$. Letting, for $1 \leqslant j \leqslant n$, $p_j = \sum_{i \in \mathbb{N}} \chi_{S_i} \otimes p_j^{(i)}$, we conclude that $p$ is $n$-majorant.

(b) is proved in a similar fashion. If $p$ is $n$-majorant, then $p(t)$ is $n$-majorant for almost every $t$, hence $p(t)$ is infinite almost everywhere. If this is the case, we can use Lemma 2.5, and assume that $p = \sum_{i \in \mathbb{N}} \chi_{S_i} \otimes p^{(i)}$, where $p^{(i)} \in \mathscr{P}(B(\ell_2))$ is infinite for each $i$, and $\cup_i S_i = \Omega$. Then $p \succ \mathbf{1} - p$.

(c) Suppose $p$ is a projection in a $\mathrm{II}_\infty$ von Neumann algebra $\mathscr{N}$. Then there exists a central projection $e$ in $p\mathscr{N}p$ such that $e\mathscr{N}e$ and $(p-e)\mathscr{N}(p-e)$ are von Neumann algebras of types $\mathrm{II}_1$ and $\mathrm{II}_\infty$, respectively. It suffices show that $e = 0$. Then, an application of Halving Lemma will complete the proof.

Denote the central cover of $e$ (in $\mathscr{N}$) by $f$. As $e$ and $p - e$ have no equivalent central projections, Proposition 6.1.8 of [9] implies that the central cover of $p - e$ is disjoint from $f$. Therefore, the central cover of $p - e$ equals $\mathbf{1} - f$ (otherwise, there exists a central projection $g \in \mathscr{N}$, disjoint from the central cover of $p$; in particular, no subprojection of $g$ is equivalent to a subprojection of $p$, which contradicts the $n$-majoration assumption).

Suppose, for the sake of contradiction, that $f \neq 0$. Write $\mathbf{1} - p = \sum_{k=1}^n p_k$, where the projections $p_k$ are mutually orthogonal, and, for each $k$, $p_k \prec p$. Then $p_k f \prec pf = e$, hence $f = \sum_{k=1}^n p_k f + e$ is finite (as a sum of finitely many finite projections). This, however, contradicts the definition of type $\mathrm{II}_\infty$.

Finally, (d) follows directly from the Halving Lemma. $\square$

### 2.2. Products of projections: sufficient conditions

We first establish two lemmas, dealing with representing "nice" operators on "small" subspaces as products of projections. The first of these lemmas essentially comes from [12].

**Lemma 2.6.** *Suppose $\mathscr{N}$ is a von Neumann algebra, $p \in \mathscr{N}$ is a projection such that $p \prec p^\perp$, and $a \in \mathbb{N}$ satisfies $0 \leqslant a \leqslant \mathbf{1}$. Then there exists $q \in \mathscr{P}(\mathscr{N})$ such that $pap = pqp$, and $q \prec p$.*

**Proof.** The construction seems to be folklore (see e.g. [1,12]), but we reproduce it here for the sake of completeness. Without loss of generality, assume that $\overline{\mathbf{ran}\, a} = \mathbf{ran}\, p$. Find a unitary $u \in \mathscr{N}$ s.t. $p_0 = u^* p u \leqslant p^\perp$. Then $q = a + (a - a^2)^{1/2} u + u^*(a - a^2)^{1/2} + u^*(\mathbf{1} - a)u$ is a projection, and $a = pqp$. Moreover, $v^* q v = p$, where $v = a^{1/2} + u^*(p - a)^{1/2} + (p - a)^{1/2} u - u^* a^{1/2} u$. $\square$

**Corollary 2.7.** *Suppose $\mathscr{N}$ is a von Neumann algebra, and $a \in \mathscr{N}$ is such that $0 \leqslant a \leqslant \mathbf{1}$, and $\overline{\mathbf{ran}\, a} \ominus \mathbf{ker}\,(\mathbf{1} - a)$ is $n$-majorated by $\mathbf{ker}\, a$. Then there exist $p_1, \ldots, p_{n+1} \in \mathscr{P}(\mathscr{N})$, such that $a = p_1 \ldots p_n p_{n+1} p_n \ldots p_1$.*

**Proof.** As noted above, we can assume that $\mathbf{ker}\,(\mathbf{1} - a) = 0$. Let $p = \mathbf{pr}(\mathbf{ran}\, a)$. By Observation (ii) in Subsection 2.1, we can assume that $\mathscr{N}$ is of type $\mathrm{I}_k$ ($k \in \mathbb{N} \cup \{\infty\}$), $\mathrm{II}_1$, $\mathrm{II}_\infty$, or III. By Lemma 2.4, if $\mathscr{N}$ is infinite, then $p \prec p^\perp$. In this case, by Lemma 2.6, $a$ is a product of 3 projections.

Now suppose $\mathscr{N}$ is a finite von Neumann algebra, and there exist mutually orthogonal projections $q_1', \ldots, q_n'$ s.t. $p = q_1' + \cdots + q_n'$, with $q_k' \prec p^\perp$ for each $k$. Then we can write $p = q_1 + \cdots + q_n$ (as sum of mutually orthogonal projections), such that, for each $i$, $q_i \sim q_i'$, and $q_i a = a q_i$. Indeed, by Theorem 1 of [8], there exists a subprojection $q_1$ of $p$, equivalent to $p_1$, s.t. $q_1 a = a q_1$. Then $(p - q_1)a = a(p - q_1)$. Moreover, by Proposition V.1.38 of [15], $p - q_1 \sim p - q_1' = q_2' + \cdots + q_n'$. By a repeated application of Theorem 1 of [8], we obtain $q_1, \ldots q_n$ with the desired properties. Note that $\mathbf{ran}\,(a q_i) \subset \mathbf{ran}\, q_i$ for any $i$, and $\overline{\mathbf{ran}\, a} = \mathbf{ran}(q_1 + \cdots + q_n)$, hence $\overline{\mathbf{ran}\,(a q_i)} = \mathbf{ran} q_i$.

Let $r_0' = p^\perp$, and define inductively the sequences of projections $(r_k)_{k=1}^n$ and $(r_k')_{k=0}^n$ as follows: on the $k$-th step, select $r_k \leqslant r_{k-1}' + q_k$ s.t. $q_k r_k q_k = a q_k$ and $r_k \sim q_k$ (this is possible, by Lemma 2.6). Let $r_k' = r_{k-1} + q_k - r_k$. By Proposition V.1.38 of [15], $r_k' \sim r_{k-1}$. Note that $r_k \leqslant p^\perp + q_1 + \cdots + q_k$, hence it is orthogonal to $q_j$ for $j > k$. Moreover, by induction we show that $r_k$ and $r_k'$ are orthogonal to $r_j$, for $j < k$.

Now let $p_1 = p$, $a_0 = a$, and, for $k \geqslant 1$,

$$p_{k+1} = r_1 + \cdots + r_k + q_{k+1} + \cdots + q_n, \quad a_k = r_1 + \cdots + r_k + a q_{k+1} + \cdots + a q_n.$$

Then $p_k a_k p_k = a_{k-1}$ for any $k \leqslant n$. Therefore, $a = p_1 \ldots p_n p_{n+1} p_n \ldots p_1$. $\square$

**Proof of Theorem 1.4.** Lemma 2.7 yields (1) $\Rightarrow$ (2). To show (2) $\Rightarrow$ (1), suppose $a = p_1 \ldots p_n p_{n+1} p_n \ldots p_1$. Without loss of generality, $p_1 \wedge \ldots \wedge p_{n+1} = \mathbf{ker}\,(\mathbf{1} - a) = \{0\}$. Let $u = p_{n+1} p_n \ldots p_1$. Note that $a = u^* u$, hence $\mathbf{ker}\, a = \mathbf{ker}\, u$, and $\overline{\mathbf{ran}\, a} = \overline{\mathbf{ran}\, u^*}$. By Proposition 6.1.6 of [9], $\overline{\mathbf{ran}\, u} \sim \overline{\mathbf{ran} u^*}$. By Lemma 2.15, $\overline{\mathbf{ran}\, u^*} \ll_n \mathbf{ker}\, u$. Therefore, $\overline{\mathbf{ran}\, a} \ll_n \mathbf{ker} a$. $\square$

**Lemma 2.8.** *Suppose $\mathscr{N}$ is a von Neumann algebra, $p \in \mathscr{P}(\mathscr{N})$, and $u \in \mathscr{N}$ is a unitary, for which $upu^* \leqslant p^\perp$. Fix $\lambda \in (0, 1)$, and let $n = \lceil \pi^2/(8(1 - \lambda)) \rceil$. Then there exist $p_1, \ldots, p_n \in \mathscr{P}(\mathscr{N})$ such that $p_n \ldots p_1 p = \lambda up$.*

**Proof.** Note that, for each $\alpha \in \mathbb{R}$, $\cos^2 \alpha = 1 - \sin^2 \alpha \geqslant 1 - \alpha^2$. Therefore, by the choice of $n$,

$$\left(\cos \frac{\pi}{2n}\right)^n \geqslant \left(1 - \frac{\pi^2}{4n^2}\right)^{n/2} \geqslant 1 - \frac{\pi^2}{4n} \geqslant \lambda.$$

Therefore, there exist $0 = \phi_0 < \phi_1 < \cdots < \phi_{n-1} < \phi_n = \pi/2$, such that $\prod_{k=1}^n \cos(\phi_k - \phi_{k-1}) = \lambda$. For $1 \leqslant k \leqslant n$, denote by $p_k$ the projection onto $\mathbf{ran}\,(\cos \phi_k p + \sin \phi_k up)$.

Pick an orthonormal basis $(\xi_i^{(0)})_{i \in \mathscr{I}}$ in $\mathbf{ran}\, p$. For $1 \leqslant k \leqslant n$, let $\xi_i^{(k)} = \cos \phi_k \xi_i^{(0)} + \sin \phi_k u \xi_i^{(0)}$. It is easy to verify that

$$\langle \xi_i^{(k)}, \xi_j^{(\ell)} \rangle = (\cos \phi_k \cos \phi_\ell + \sin \phi_k \sin \phi_\ell)\delta_{ij} = \cos(\phi_k - \phi_\ell)\delta_{ij}$$

(here $\delta_{ij}$ is Kronecker's delta). In particular, $p_k \xi_i^{(k-1)} = \cos(\phi_k - \phi_{k-1})\xi_i^{(k)}$. Therefore,

$$p_n \cdots p_1 \xi_i^{(0)} = \prod_{k=1}^n \cos(\phi_k - \phi_\ell)\xi_i^{(n)},$$

and we are done. $\square$

**Lemma 2.9.** *Suppose $\mathscr{N}$ is a von Neumann algebra, $p \in \mathscr{N}$ is a projection, such that $p \prec p^\perp$. Suppose, furthermore, that $u \in \mathscr{N}$ is a unitary, and $u(\mathbf{ran}\, p) \subset \mathbf{ker} p$. Then, for any $\lambda \in (0, 1)$, $\mathbf{M}(\lambda up) \leqslant 2\lceil \pi^2/(8(1 - \sqrt{\lambda})) \rceil$.*

**Proof.** Find a unitary $v \in \mathscr{N}$ such that $q = vpv^* \leqslant p^\perp$. Apply Lemma 2.8 twice-first to $\sqrt{\lambda} vp$, then to $\sqrt{\lambda} uv^* q$. $\square$

**Corollary 2.10.** *Suppose $\mathscr{N}$ is a von Neumann algebra, $p \in \mathscr{P}(\mathscr{N})$, and $p^\perp$ is n-majorant. Suppose, furthermore, that $u \in \mathscr{N}$ is a unitary, and $u(\mathbf{ran}\, p) = \mathbf{ran}\, p$. Then, for any $\lambda \in (0, 1)$, $\mathbf{M}(\lambda up) \leqslant 2n\lceil \pi^2/(8(1 - \sqrt{\lambda})) \rceil$.*

**Proof.** As before, we can consider the cases of $\mathscr{N}$ being infinite, and $\mathscr{N}$ being finite, separately. If $\mathscr{N}$ is infinite, then $p \prec p^\perp$ by Lemma 2.4, and an application of Lemma 2.9 completes the proof. If $\mathscr{N}$ is finite, use Theorem 1 of [8] to write $p$ as a sum of mutually orthogonal projections $p_1, \ldots, p_n$, s.t. $p_k \prec p^\perp$ for each $k$, and $up_k = p_k u$. Then $pu = \prod_{k=1}^n (p_k u + p - p_k)$, and, by Lemma 2.9, $\mathbf{M}(p_k u + p - p_k) \leqslant 2\lceil \pi^2/(8(1 - \sqrt{\lambda})) \rceil$. $\square$

**Proof of Theorem 1.3.** By [12], $\sigma(a) \subset (-1, 1]$ whenever $a$ is a self-adjoint product of projections. Suppose $\sigma(a) \subset [\lambda, 1]$. Consider a function

$$\phi(x) = \begin{cases} x & x \geqslant 0 \\ 0 & x < 0 \end{cases},$$

and let $a_+ = \phi(a)$, $a_- = a - a_+$. By the spectral theorem, $p_+ = \overline{\mathbf{ran}\, a_+}$ and $p_- = \overline{\mathbf{ran}\, a_-}$ are mutually orthogonal. By Corollary 2.7, $a_+$ (viewed as acting on $p_+ + p^\perp$) is a product of $2n+1$ projections. By Theorem 1.1(1), $a_-$ (viewed as acting on $p_- + p^\perp$) is a product of $\gamma n/(1-\lambda)$ projections. This yields an upper estimate for $\mathbf{M}(a)$. □

**Lemma 2.11.** *Suppose $p, q \in \mathscr{P}(\mathscr{N})$ are such that $p \sim q$ and $p^\perp \sim q^\perp$. Fix $\lambda \in (0, 1)$, and let $n = \lceil 2\pi^2/(1-\lambda)\rceil$. Then there exist $p_1, \ldots, p_n \in \mathscr{P}(\mathscr{N})$ such that $p_n \ldots p_1(\mathbf{ran}\, p) = \mathbf{ran} q$, and $\|p_n \ldots p_1 \xi\| \geqslant \lambda \|\xi\|$ for any $\xi \in \mathbf{ran}\, p$.*

**Proof.** Find partial isometries $v, w \in \mathscr{N}$ s.t. $v^*v = p$, $vv^* = q$, $w^*w = p^\perp$, and $ww^* = q^\perp$. Then $u = v + w$ is a unitary. Write $u = \exp(ia)$, where $a \in \mathscr{N}$ is self-adjoint, with $\|a\| \leqslant \pi$. Let $n = \lceil 2\pi^2/(1-\lambda)\rceil$. For $0 \leqslant k \leqslant n$, denote by $p_k$ the projection onto $\mathbf{ran}(\exp(ika/n)p)$ (in this notation, $p_0 = p$, and $p_n = q$). For $1 \leqslant k \leqslant n$, $\|p_k - p_{k-1}\| \leqslant \|\exp(ia/n) - \mathbf{1}\| \leqslant \|a\|/n \leqslant \pi/n$. Therefore, for every norm 1 $\xi \in \mathbf{ran}\, p_{k-1}$, $\|p_k\xi - \xi\| \leqslant \pi/n$. Thus, $\|p_k\xi\|^2 = \|\xi\|^2 - \|p_k\xi - \xi\|^2 \geqslant 1 - \pi^2/n^2$. Therefore,

$$\|p_n \ldots p_1\xi\| \geqslant (1 - \pi^2/n^2)^{n/2}\|\xi\| \geqslant (1 - \pi^2/(2n))\|\xi\| \geqslant \lambda\|\xi\|$$

for any $\xi \in \mathbf{ran}\, p$. □

**Remark 2.12.** For a von Neumann algebra $\mathscr{N}$ and $p \in \mathscr{P}(\mathscr{N})$, denote by $\mathscr{S}(p)$ the set of all $q \in \mathscr{P}(\mathscr{N})$ s.t. $p \sim q$ and $p^\perp \sim q^\perp$. The above proof shows that $\mathscr{S}(p)$ is path-connected, and moreover, any such $p$ and $q$ can be connected by a path of length at most $\pi$. See [4] for more on this topic.

**Proof of Theorem 1.1(1).** By Section 2.1, it suffices to consider the case of $\|a\| < 1$. Denote the initial and terminal projections of $a$ by $p$ and $q$, respectively. Let $\lambda = \|a\|^{1/2}$. By Lemma 2.11, there exists $b \in \mathscr{N}$ s.t. for any $\xi \in \mathbf{ran}\, p$, we have $bp \in \mathbf{ran}\, q$, $\|bp\| \geqslant \lambda\|\xi\|$, and $\mathbf{M}(b) \leqslant \lceil 2\pi^2/(1-\lambda)\rceil \leqslant \lceil 4\pi^2/(1-\|a\|)\rceil$. As $p^\perp \sim q^\perp$, there exists a partial isometry $v$ s.t. $vv^* = p^\perp$ and $v^*v = q^\perp$. Then $a' = (bp + v)^{-1}a \in \mathscr{N}$ fixes $\mathbf{ran}\, p$, $ba' = a$, and $\|a'\| \leqslant \|(b|_{\mathbf{ran}\, p})^{-1}\|\,\|a\| \leqslant \lambda$.

Write $a' = \|a'\|up \cdot \|a'\|^{-1}|a'|$, where $u \in \mathscr{N}$ is a unitary, fixing $\mathbf{ran}\, p$. By Lemma 2.7, $\mathbf{M}(\|a'\|^{-1}|a'|) \leqslant 2n + 1$. By Lemma 2.10,

$$\mathbf{M}(\|a'\|up) \leqslant 2n\lceil \pi^2/(8(1 - \|a'\|^{1/2}))\rceil \leqslant 2n\lceil \pi^2/(2(1-\|a\|))\rceil.$$

Together, the estimates on $\mathbf{M}(b)$, $\mathbf{M}(\|a'\|up)$, and $\mathbf{M}(\|a'\|^{-1}|a'|)$ yield an upper estimate on $\mathbf{M}(a)$. □

### *2.3. Necessary conditions and lower bounds*

**Lemma 2.13.** *Suppose $p_1, \ldots, p_n$ are projections in a von Neumann algebra $\mathscr{N}$. Then $\mathbf{ran}\, (p_n \ldots p_1)^\perp$ is equivalent to $\mathbf{ker}\, (p_n \ldots p_1)$.*

**Corollary 2.14.** *Suppose $p_1, \ldots, p_n \in \mathscr{P}(\mathscr{N})$. Then there exists a unitary $u \in \mathscr{N}$, mapping $\mathbf{ker}(p_n \ldots p_1)$ onto $\mathbf{ran}(p_n \ldots p_1)^\perp$, and $\mathbf{ker}(p_n \ldots p_1)^\perp$ onto $\overline{\mathbf{ran}(p_n \ldots p_1)}$.*

**Proof.** By Proposition 6.1.6 of [9],

$$\overline{\mathbf{ran}(p_n \dots p_1)} \sim \overline{\mathbf{ran}((p_n \dots p_1)^*)} = \mathbf{ker}(p_n \dots p_1)^\perp.$$

An application of the previous lemma finishes the proof. □

**Proof of Lemma 2.13.** This statement can be proved by induction over $n$. The case of $n = 1$ is trivial. We handle the case of $n = 2$ by showing that, for any two projections $p, q \in \mathcal{N}$, $\mathbf{ker}\,(pq)$ is equivalent to $(\mathbf{ran}(pq))^\perp$. By passing to the orthogonal complement of $p \wedge q$ if necessary, we can assume that $p \wedge q = 0$. Let $p_1 = \mathbf{pr}(\mathbf{ran}(pq)) = p - p \wedge q^\perp$ (cf. Proposition 2.5.14 of [9]), and $q_1 = \mathbf{pr}(\mathbf{ker}(pq))^\perp = \mathbf{pr}(\mathbf{ran}(qp)) = q - q \wedge p^\perp$. Then $p_1$ is injective on $\mathbf{ran}\, q_1$, and $\mathbf{ran}(p_1 q_1) = \mathbf{ran}(pq)$ is dense in $\mathbf{ran}\, p_1$. We show that

$$\overline{p_1^\perp(\mathbf{ran}\, q_1^\perp)} = \mathbf{ran}\, p_1^\perp. \tag{2.1}$$

Indeed, suppose $\xi \in \mathbf{ran}\, p_1^\perp$ is orthogonal to $p_1^\perp q_1^\perp \eta$ for any $\eta$. Then

$$\langle q_1^\perp \eta, \xi\rangle = \langle q_1^\perp \eta, p_1^\perp \xi\rangle = \langle p_1^\perp q_1^\perp \eta, \xi\rangle = 0,$$

hence $\xi \in \mathbf{ran}\, q_1$, and (2.1) follows. Therefore, $p_1^\perp \prec q_1^\perp$. Similarly, we show that $p_1^\perp \succ q_1^\perp$. Thus, $p_1^\perp \sim q_1^\perp$.

Now suppose the statement is true for $n$, and $p_1, p_2, \dots, p_n, p_{n+1}$ are projections in $\mathcal{N}$. By Lemma 2.1, we can assume that, for $1 \leqslant j \leqslant n+1$, $\mathbf{ker}\,(p_j \dots p_1) = \mathbf{ker}\, p_1$ (hence, $\mathbf{ker}\, p_j \cap \mathbf{ran}\,(p_{j-1} \dots p_1) = \{0\}$), and $\mathbf{ran}\, p_j = \overline{\mathbf{ran}(p_j \dots p_1)}$. Let $r = p_n \wedge p_{n+1}^\perp$, and $p'_n = p_n r^\perp$. By assumption, $\mathbf{ker}\, p_{n+1} \cap \mathbf{ran}(p_n \dots p_1) = \emptyset$, hence

$$\mathbf{ker}(p'_n p_{n-1} \dots p_1) = \mathbf{ker}\,(p'_n p_n p_{n-1} \dots p_1) = \mathbf{ker}(p_n \dots p_1) = \mathbf{ker}\, p_1.$$

Note that $\overline{\mathbf{ran}(p'_n p_{n-1} \dots p_1)} = \mathbf{ran}\, p'_n$. By the induction hypothesis, $\mathbf{ker}\, p'_n \sim \mathbf{ker}\, p_1$. Moreover, $p_{n+1}(p_n - p'_n) = 0$, hence $p_{n+1} p_n \dots p_1 = p_{n+1} p'_n p_{n-1} \dots p_1$. In particular, $\overline{\mathbf{ran}(p_{n+1} p'_n)} = \mathbf{ran}\, p_{n+1}$. Finally, $p_{n+1}|_{\mathbf{ran}\, p'_n}$ is surjective, hence $\mathbf{ker}\,(p_{n+1} p'_n) = \mathbf{ker}\, p'_n$. However, as shown above, $\mathbf{ran}(p_{n+1} p'_n)^\perp \sim \mathbf{ker}\,(p_{n+1} p'_n)$. Taken together, all the equivalences of the last paragraph complete the proof. □

**Lemma 2.15.** *If $n \geqslant 2$, and $p_1, \dots, p_n \in \mathcal{P}(\mathcal{N})$ are such that $p_1 \wedge \dots \wedge p_n = \{0\}$, then* $\overline{\mathbf{ran}(p_n \dots p_1)} \ll_{n-1} \mathbf{ker}\,(p_n \dots p_1)$.

**Proof.** By Lemma 2.1, we assume that, for each $k$, $\mathbf{ker}\, p_k \cap \mathbf{ran}\,(p_{k-1} \dots p_1) = \{0\}$ (hence $\mathbf{ker}\,(p_k \dots p_1) = \mathbf{ker}\, p_1$), and $\mathbf{ran}\, p_k = \overline{\mathbf{ran}(p_k \dots p_1)}$.

The lemma is proved by induction over $n$. If $n = 2$, we have to show that, given $p_1, p_2 \in \mathcal{P}(\mathcal{N})$ with $p_1 \wedge p_2 = \{0\}$, we have $\overline{\mathbf{ran}(p_2 p_1)} \prec \mathbf{ker}(p_2 p_1)$. By Kaplansky Formula,

$$\begin{aligned}\mathbf{pr}(\mathbf{ker}\,(p_2 p_1)) &= \mathbf{pr}(\mathbf{ker}\, p_1) = (p_1 \vee p_2)^\perp + (p_1 \vee p_2 - p_1)\\ &\sim (p_1 \vee p_2)^\perp + p_2 \geqslant p_2 = \mathbf{pr}(\mathbf{ran}(p_2 p_1)).\end{aligned}$$

Now suppose the statement of the lemma holds for $n$, and prove it for $n+1$. By Lemma 2.13, $\mathbf{ker}\, p_k \sim \mathbf{ker}\, p_1$ for $2 \leqslant k \leqslant n+1$. Let $r = p_1 \wedge \dots \wedge p_n$, and $q = \mathbf{pr}(\mathbf{ran}(p_n \dots p_1)) - r$. Clearly, $r$ commutes with $p_k$ $(1 \leqslant k \leqslant n)$. Applying the induction hypothesis to the projections $p_k r^\perp$ $(1 \leqslant k \leqslant n)$, we conclude that $q \ll_{n-1} r^\perp - r^\perp p_1 \leqslant p_1^\perp$. Moreover, $r \wedge p_{n+1} = \{0\}$, hence $\mathbf{ker}(p_{n+1}^\perp r) = \{0\}$. This implies $r \prec p_{n+1}^\perp \sim p_1^\perp$. Thus, by Lemma 2.2,

$$\mathbf{pr}(\mathbf{ran}(p_{n+1} \dots p_1)) \leqslant \mathbf{pr}(\mathbf{ran}(p_{n+1}(q + r))) \prec q + r \ll_n p_1^\perp. \qquad \square$$

**Corollary 2.16.** *If* $p_1, \ldots, p_n \in \mathscr{P}(\mathscr{N})$, *then* $\overline{\mathbf{ran}(p_n \ldots p_1)} \ominus (\wedge_{k=1}^n p_k) \ll_{n-1} \mathbf{ker}(p_n \ldots p_1)$.

This result can be reformulated as follows: suppose $a = p_n \ldots p_1$. Then $\overline{\mathbf{ran}\, a} \ominus \mathbf{ker}\,(1 - a) \ll_{n-1} \mathbf{ker}\, a$.

**Proof.** Note that the projection $p = (\wedge_{k=1}^n p_k)^\perp$ commutes with $p_k$ for each $k$. Let $p_k' = pp_k$. Then $\wedge_{k=1}^n p_k' = \{0\}$, $\mathbf{ker}\,(p_n \ldots p_1) = \mathbf{ker}\,(p_n' \ldots p_1')$, and $\overline{\mathbf{ran}(p_n \ldots p_1)} \ominus (\wedge_{k=1}^n p_k) = \overline{\mathbf{ran}(p_n' \ldots p_1')}$. An application of Lemma 2.15 completes the proof. $\square$

**Proof of Theorem 1.1(2).** Combine Corollaries 2.14 and 2.16. $\square$

**Remark 2.17.** The conditions $\|u\| < 1$, $\mathbf{ker}\, u \sim (\mathbf{ran}\, u)^\perp$ are not sufficient to guarantee that $u$ is a product of projections. Indeed, suppose $\mathscr{N} = \ell_\infty(\mathscr{R})$, where $\mathcal{R}$ is a $\mathrm{II}_1$ factor. Fix $\lambda \in (0, 1)$. By Theorem 1.2, for every $n \in \mathbb{N}$ there exists $p_n \in \mathscr{P}(\mathscr{R})$ s.t. $-\lambda p_n$ cannot be represented as a product of less than $n$ projections. Consider $u = (u_n)_{n\in\mathbb{N}}$. Clearly, $\|u\| = \lambda$, and $\mathbf{ker}\, u \sim (\mathbf{ran}\, u)^\perp$, yet $u$ is not a product of projections.

**Proof of Proposition 1.2.** (1) Suppose $-\lambda p = p_n \ldots p_1$. Pick a norm one $\xi \in \mathbf{ran}\, p$. For $1 \leqslant k \leqslant n$, let $u_0 = p$, and $u_k = p_k \ldots p_1$. Note that $u_{k-1} - u_k = p_k^\perp p_{k-1} \ldots p_1$, hence $\|(u_{k-1} - u_k)\xi\|^2 = \alpha_k^2$, where $\alpha_k = (\|u_{k-1}\xi\|^2 - \|u_k\xi\|^2)^{1/2}$. But

$$1 - \lambda^2 = \|\xi\|^2 - \|p_n \ldots p_1\xi\|^2 = \sum_{k=1}^n \alpha_k^2.$$

Thus,

$$1 + \lambda = \| \sum_{k=1}^n (u_{k-1} - u_k)\xi \| \leqslant \sum_{k=1}^n \alpha_k \leqslant \sqrt{n} \left( \sum_{k=1}^n \alpha_k^2 \right)^{1/2} = \sqrt{n}\sqrt{1 - \lambda^2},$$

hence $n \geqslant (1 + \lambda)/(1 - \lambda)$.

Now suppose $-\lambda p = p_k \ldots p_1$, and show that $p^\perp$ is $k$-majorant. Clearly, $p_1 \wedge \cdots \wedge p_k = \{0\}$. For notational simplicity, let $p_0' = p_0 = p$. For $1 \leqslant i \leqslant k$, define $p_i' = p \wedge p_1 \wedge \cdots \wedge p_i$, and $q_i = p_{i-1}' - p_i'$. Note that the projections $q_i$ are mutually orthogonal, and $p = \sum_{i=1}^k q_i$. Moreover, for each $i$, $p_i\xi \neq \xi$ for any $\xi \in \mathbf{ran}\, q_i$ (otherwise, we would have $\xi \in \mathbf{ran}\,(p \wedge p_1 \wedge \ldots \wedge p_{i-1} \wedge p_i)$), hence $p^\perp p_i|_{\mathbf{ran}\, q_i}$ is injective. Thus, $q_i$ is equivalent to $\mathbf{pr}(p^\perp p_i q_i)$, and therefore, $p^\perp$ is $k$-majorant.

(2) Suppose $p$ is as in the statement of the theorem, and $-\lambda p = p_n \ldots p_1 = p_n \ldots p_1 p$. As above, let $u_0 = p$, and $u_k = p_k \ldots p_1 p$ for $1 \leqslant k \leqslant n$. For such $k$, let $v_k = u_{k-1} - u_k = p_k^\perp p_{k-1} \ldots p_1 p$. Note that $p \prec p_k$ for each $k$, hence $\tau(p_k^\perp) = 1 - \tau(p_k) \geqslant 1 - \tau(p)$.

Denote the matrix units in $M_n$ by $E_{ij}$ ($1 \leqslant i, j \leqslant n$). Let $M_n(\mathscr{N})$ be the von Neumann algebra of all $n \times n$ matrices with entries in $\mathscr{N}$. Equip $M_n(\mathscr{N})$ with the trace $\tau_n$ defined by $\tau_n(\sum_{i,j=1}^n E_{ij} \otimes a_{ij}) = \sum_{i=1}^n \tau(a_{ii})$. We identify $\mathscr{N}$ with $E_{11} \otimes \mathscr{N} \hookrightarrow M_n(\mathscr{N})$ (that is, with the upper left corner of $M_n(\mathscr{N})$)). Define $v, w \in M_n(\mathscr{N})$ by setting $v = \sum_{i=1}^n E_{i1} \otimes v_i$, and $w = \sum_{i=1}^n E_{1i} \otimes p_i^\perp$. Then $(1 + \lambda)p = wv$, hence $\|(1 + \lambda)p\|_2 \leqslant \|w\|_2\|v\|$ (here, $\|a\|_2 = (\tau_n(a^*a))^{1/2}$ for $a \in M_n(\mathscr{N})$). But $\|(1 + \lambda)p\|_2 = (1 + \lambda)\tau(p)^{1/2}$. Moreover,

$$ww^* = \sum_{k=1}^n p_k^\perp p_{k-1} \ldots p_1 p p_1 \ldots p_{k-1} p_k^\perp \leqslant \sum_{k=1}^n p_k^\perp,$$

hence

$$\|w\|_2^2 = \tau_n(w^*w) = \tau(ww^*) \leqslant \sum_{k=1}^{n} \tau(p_k^\perp) \leqslant n(1-\tau(p)).$$

Finally, $\|v\| \leqslant \sqrt{1-\lambda^2}$. To see this, view $\mathscr{N}$ as acting on a Hilbert space $H$. If $\xi \in \mathbf{ran}\, p$, then the reasoning of part (1) shows that

$$\|v\xi\|^2 = \sum_i \|(u_{i-1}-u_i)\xi\|^2 = \sum_i (\|u_{i-1}\xi\|^2 - \|u_i\xi\|^2) = (1-\lambda^2)\|\xi\|^2.$$

As $v = v(\sum_i E_{ii} \otimes p)$, we obtain the desired estimate for $\|v\|$.

By the above inequalities, $(1+\lambda)^2\tau(p) \leqslant n(1-\tau(p))(1-\lambda^2)$, which yields the lower estimate for $n$. $\square$

### 2.4. Closure of products of projections

For the proof of Theorem 1.5, we need a technical lemma (it may be known to specialists).

**Lemma 2.18.** *Suppose $\mathscr{N}$ is a separably acting von Neumann algebra without summands of finite type* I. *Then there exists a double indexed sequences of projections $p_n^{(k)}$ in $\mathscr{N}$ ($k \in \mathbb{N}$, $1 \leqslant n \leqslant 2^k$) such that*:

(1) *For each $k$, the projections $(p_n^{(k)})_{n=1}^{2^k}$ are mutually orthogonal, equivalent to each other, and $\sum_{n=1}^{2^k} p_n^{(k)} = \mathbf{1}$. If $\mathscr{N}$ is properly infinite, then $p_n^{(k)} \sim \mathbf{1}$ for any $n$ or $k$.*
(2) *The sequence $(p_{2^k}^{(k)})_{k\in\mathbb{N}}$ converges to 0 in the strong operator topology.*

**Proof.** By the type decomposition, we can assume that either $\mathscr{N}$ is finite (hence of type $\mathrm{II}_1$), or it is properly infinite. Part (1) can be obtained by a repeated applications of the standard "halving" results (see e.g. Propositions V.1.35 and V.1.36 of [15]). Moreover, we have $p_n^{(k)} = p_{2n-1}^{(k+1)} + p_{2n}^{(k+1)}$ for any $1 \leqslant n \leqslant 2^k$.

To tackle (2), recall that $\mathscr{N}$ is a WOT closed subalgebra of $B(H)$, where $H$ is a separable Hilbert space. Suppose $(\xi_i)_{i\in\mathbb{N}}$ is a dense subset of the unit sphere $H$. It suffices to show that there exists a sequence $(n_k)_{k\in\mathbb{N}}$ such that $p_{n_k}^{(k)}\xi_i \to 0$ for every $i$. More precisely, we shall select sequences $(n_k)$ and $(k_s)$ so that $\|p_{n_s}^{(k_s)}\xi_i\| \leqslant 2^{-s}$ whenever $1 \leqslant i \leqslant s$.

By the Pythagorean theorem,

$$\sum_{n=1}^{2^k} \|p_n^{(k)}\xi_i\|^2 = 1 \tag{2.2}$$

for each $i$ and $k$. More generally, for any $i, k > m$, and $1 \leqslant \ell \leqslant 2^m$,

$$\sum_{n=2^{k-m}(\ell-1)+1}^{2^{k-m}\ell} \|p_n^{(k)}\xi_i\|^2 = \|p_\ell^{(m)}\xi_i\|^2. \tag{2.3}$$

We select $k_1$ and $n_1$ using (2.2). Suppose $n_1, \ldots, n_s, k_1, \ldots, k_s$ satisfying our inequalities have been selected. Let $k_{s+1} = k_s + s + 2$, and find $n_{s+1} \in (2^{k_{s+1}-k_s}n_s, 2^{k_{s+1}-k_s}n_s]$ with the desired properties. To this end, for $1 \leqslant i \leqslant s+1$, let

$$S_i = \{n \in (2^{k_{s+1}-k_s} n_s, 2^{k_{s+1}} n_s] | \|p_n^{(k)} \xi_i\| \leqslant 2^{-(s+1)}\},$$

and $S_i^c = (2^{k_{s+1}-k_s} n_s, 2^{k_{s+1}-k_s} n_s] \backslash S_i$. By (2.3),

$$\sum_{n=2^{k-k_s}(n_s-1)+1}^{2^{k-k_s} n_s} \|p_n^{(k)} \xi_i\|^2 \leqslant 2^{-2s}$$

for $1 \leqslant i \leqslant s$, and

$$\sum_{n=2^{k-k_s}(n_s-1)+1}^{2^{k-k_s} n_s} \|p_n^{(k)} \xi_{s+1}\|^2 \leqslant 1.$$

Therefore, $|S_i^c| < 4$ for $1 \leqslant i \leqslant s$, and $|S_{s+1}^c| < 2^{2(s+1)}$. But

$$|(2^{k_{s+1}-k_s} n_s, 2^{k_{s+1}-k_s} n_s]| = 2^{k_{s+1}-k_s} > 4^{s+1} + 4s > \sum_{i=1}^{s+1} |S_i^c|,$$

and therefore, by the pigeon-hole principle, there exists $n_s \in \cup_{i=1}^{s+1} S_i$. $\square$

**Proof of Theorem 1.5.** First suppose $\mathcal{N}$ has no summands of type $\mathrm{I}_n$. It suffices to show that any $u \in \mathcal{N}$ with $\|u\| < 1$ belongs to the $\boldsymbol{\tau}$-closure of the products of projections.

Write $\mathcal{N} = \mathcal{N}_{[f]} \oplus \mathcal{N}_{[i]}$, where the von Neumann algebras $\mathcal{N}_{[f]}$ and $\mathcal{N}_{[i]}$ are finite and properly infinite, respectively. The identities of the summands are denoted by $\mathbf{1}_{[i]}$ and $\mathbf{1}_{[f]}$, respectively. By Lemma 2.18, there exist mutually orthogonal projections $(p_{[f],n}^{(k)})_{n=1}^{2^k} \in \mathcal{N}_{[f]}$ and $(p_{[i],n}^{(k)})_{n=1}^{2^k} \in \mathcal{N}_{[f]}$, so that

(1) $p_{[f],m}^{(k)} \sim p_{[f],n}^{(k)}$ if $m \neq n$, and $p_{[i],n}^{(k)} \sim \mathbf{1}_{[i]}$ for any $n$;
(2) $\sum_{n=1}^{2^k} p_{[i],n}^{(k)} = \mathbf{1}_{[i]}$, and $\sum_{n=1}^{2^k} p_{[f],n}^{(k)} = \mathbf{1}_{[f]}$;
(3) the sequences $(p_{[i],2^k}^{(k)})_{k\in\mathbb{N}}$ and $(p_{[f],2^k}^{(k)})_{k\in\mathbb{N}}$ converge to 0 in the SOT.

For $k \in \mathbb{N}$ and $\alpha \in \{i, f\}$, let $v_{[\alpha],k} = p_{[\alpha],2^k}^{(k)\perp} u p_{[\alpha],2^k}^{(k)\perp}$. Clearly, $v_{[i],k} + v_{[f],k} \to u$ in the topology $\boldsymbol{\tau}$. It remains to approximate $v_{[\alpha],k}$ in the norm topology by products of projections.

Denote the initial and final projections of $v_{[\alpha],k}$ by $\overline{p}_{[\alpha],k}^{(\mathrm{in})}$ and $\overline{p}_{[\alpha],k}^{(\mathrm{fi})}$, respectively.

First consider $v_{[f],k}$. By Proposition V.1.38 of [15], $\mathbf{1}_{[f]} - \overline{p}_{[\alpha],k}^{(\mathrm{in})} \sim \mathbf{1}_{[f]} - \overline{p}_{[\alpha],k}^{(\mathrm{fi})}$. That is, $\mathbf{ker}\, v_{[i],k} \sim (\mathbf{ran}\, v_{[i],k})^{\perp}$. Moreover, $\mathbf{ran}\, p_{2^k}^{(k)} \hookrightarrow \mathbf{ker}\, v_{[i],k}$, hence $\mathbf{ker}\, v_{[i],k}$ is $2^k$-majorant. By Theorem 1.1, $v_{[f],k}$ is a product of projections.

Now consider $v_{[i],k}$. Then $\mathbf{ker}\, v_{[i],k} \succ p_{[i],2^k}^{(k)} \sim \mathbf{1}_{[i]}$, hence $\mathbf{ker}\, v_{[i],k}$ is 1-majorant. Moreover, $(\mathbf{ran}\, v_{[i],k})^{\perp} \succ p_{[i],2^k}^{(k)} \sim \mathbf{1}_{[i]}$, hence $\mathbf{ker}\, v_{[i],k} \sim (\mathbf{ran}\, v_{[i],k})^{\perp}$. By Theorem 1.1, $v_{[i],k}$ is a product of projections.

Now suppose $z\mathcal{N}$ is a type $\mathrm{I}_n$ von Neumann algebra for some central projection $z \in \mathcal{N}$ and $n \in \mathbb{N}$. It suffices to show that $-z$ does not belong to the WOT closure of the products of projections in $z\mathcal{N}$. We shall find a WOT continuous linear functional $f$ on $z\mathcal{N}$ s.t. $f(z) = n$, and, for any product of projections $u$, either $|f(u)| \leqslant n - 1$, or $u = z$.

By Theorems 9.3.2 and 9.4.1 of [9], $z\mathcal{N}$ is unitarily equivalent (via a unitary $U$) to $M_n \otimes L_\infty(\Omega, \mu) \otimes I_K$, where $K$ is a Hilbert space, and $\mu$ is a $\sigma$-finite measure on a set $\Omega$. Denote by

$(e_i)_{i=1}^n$ the canonical basis in $\ell_2^n$, and fix norm one vectors $\xi \in L_2(\mu)$ and $\eta \in K$. Define $f$ by setting, for $a \in z\mathscr{N}$,

$$f(a) = \sum_{i=1}^{n} \langle (U^*aU)e_i \otimes \xi \otimes \eta, e_i \otimes \xi \otimes \eta \rangle.$$

Clearly, $f(z) = n$. Moreover, if $u \in z\mathscr{N}$ is a product of projections, different from $z$, then $(U^*uU)(\omega)|_{\ell_2^n \otimes \mathbb{C}\eta}$ has norm not exceeding 1, and rank less than $n$, for almost every $\omega \in \Omega$. Then, $|f(u)| \leqslant n - 1$. $\square$

## 3. Products of idempotents

In this section, we consider products of idempotents in von Neumann algebras. A subspace $F$, affiliated with a von Neumann algebra $\mathscr{N}$, is called a *complement* of $E$ if there exists an idempotent $p \in \mathscr{N}$ s.t. $\mathbf{ran}\, p = E$ and $\mathbf{ker}\, p = F$. A complement need not be unique. However, all the complements of a given $E$ are equivalent to each other. Indeed, suppose $F_1$ and $F_2$ are two complements of $E$, corresponding to the idempotents $p_1$ and $p_2$, respectively. Then $\mathbf{1} - p_1$ is a bijection from $F_2$ to $F_1$.

Moreover (see e.g. [3,10]), if $q$ is an idempotent in a $C^*$-algebra (not necessarily a von Neumann algebra) $\mathscr{N}$, then there exists a (unique) projection $p \in \mathscr{N}$ onto the range of $q$ (equivalently, $pq = q$ and $qp = p$). Furthermore, there exists an invertible $u \in \mathscr{N}$ s.t. $q = pu$ and $up = p$. Indeed, consider

$$u = q + p^{\perp}(\mathbf{1} - q) = p^{\perp} + q = \mathbf{1} + qp^{\perp}(\mathbf{1} - q).$$

Then $u$ is invertible ($u^{-1} = \mathbf{1} - qp^{\perp}(\mathbf{1} - q)$), and the above inequalities can be easily verified. In fact, $u$ maps $\mathbf{ker}\, q$ onto $\mathbf{ker}\, p$ ($u(\mathbf{1} - q) = p^{\perp}$).

Extending this to a product of several idempotents, we obtain:

**Lemma 3.1.** *Suppose $q_1, \ldots, q_n$ are idempotents in a $C^*$-algebra $\mathscr{N}$. Then there exist projections $p_1, \ldots, p_n \in \mathscr{N}$, and an invertible element $v \in \mathscr{N}$, such that $\cap_{k=1}^n \mathbf{ran}\, p_k = \cap_{k=1}^n \mathbf{ran}\, q_k$, $q_n \ldots q_1 = vp_n \ldots p_1$, and $v|_{\cap_{k=1}^n \mathbf{ran}\, p_k} = I_{\cap_{k=1}^n \mathbf{ran}\, p_k}$.*

**Proof.** Proceed by induction over $n$. The base of induction (the case of $n = 1$) has already been established. Suppose the statement is true for $n - 1$, and prove it for $n$. Suppose $q_1, \ldots, q_n$ are idempotents. Then there exist projections $p_1, \ldots, p_{n-1}$ and an invertible $u$ s.t. $q_{n-1} \ldots q_1 = up_{n-1} \ldots p_1$, and the set of fixed points of $u$ contains $\cap_{k=1}^{n-1} \mathbf{ran}\, p_k = \cap_{k=1}^{n-1} \mathbf{ran}\, q_k$. Write $u^{-1}q_n u = wp_n$, where the projection $p_n$ has the same range as $u^{-1}q_n u$, $w$ is invertible, and $w|_{\mathbf{ran}\, p_n} = I_{\mathbf{ran}\, p_n}$. Therefore, $q_n \ldots q_1 = vp_n \ldots p_1$, with $v = uw$, has the desired properties. $\square$

**Proof of Theorem 1.6(1).** Suppose $a$ is a product of $n$ idempotents, belonging to a von Neumann algebra $\mathscr{N}$, acting on a separable Hilbert space $H$. By Lemma 3.1, we can write $a = vp_n \ldots p_1$, where $v$ is invertible, and $p_1, \ldots, p_n \in \mathscr{P}(\mathscr{N})$ s.t. $a = vp_n \ldots p_1$. Moreover, $v$ fixes $E = \cap_{k=1}^n \mathbf{ran}\, p_k$. Let $F = H \ominus (E + \mathbf{ker}\, u)$ (clearly, $E + \mathbf{ker}\, u$ is closed). Then $F' = \overline{p_n \ldots p_1(F)}$ is orthogonal to $E$ (cf. Observation (iii) of Section 2.1). Let $G = v(F')$. As $v$ is invertible, $G$ and $E$ are at positive angles. An application of Lemma 2.13 and Corollary 2.16 completes the proof. $\square$

To prove Theorem 1.6(2), we need some auxiliary results.

**Lemma 3.2.** *Suppose $\mathcal{N}$ is a von Neumann algebra, and the projections $p_1, p_2 \in \mathcal{N}$ satisfy $\|p_1 p_2\| < 1$. Then for any $u \in \mathcal{N}$ satisfying $u = p_1 u p_2$ there exists an idempotent $h \in \mathcal{N}$ such that $hp_2 = u$, and $p_1 h = h$.*

**Proof.** By the proof of Theorem 2.1 of [10], there exists an idempotent $q \in \mathcal{N}$ s.t. $qp_1 = p_1$, $p_1 q = q$, and $qp_2 = 0$ (in other words, $\mathbf{ran}\, q = \mathbf{ran}\, p_1$ and $\mathbf{ker}\, q \supset \mathbf{ran}\, p_2$). Then $h = qu(\mathbf{1} - q) + q$ is an idempotent ($h^2 = h$). Moreover, $qp_2 = 0$, hence $(\mathbf{1} - q)p_2 = p_2$, and $hp_2 = qup_2 = qp_1 up_2 = p_1 up_2 = u$. Finally, $p_1 q = q$, hence $p_1 h = h$. $\square$

**Proof of Theorem 1.6(2).** Consider $u$ and $F$ as in the statement of the theorem. Then the spaces $G = \overline{u(F)}$ and $G' = \mathbf{pr}(E)^{\perp}(G)$ are affiliated with $\mathcal{N}$. Let $p_0$, $p$ and $q$ be the projections onto $E$, $G'$, and $(E + G)^{\perp}$, respectively. Moreover, let $p'$ be the idempotent whose range is $E$, and whose kernel is $G + (E + G)^{\perp}$ (it belongs to $\mathcal{N}$, by Theorem 2.1 of [10]). Note that $\mathbf{ker}\, p = \mathbf{ker}\, p'$, hence, by the remark before Lemma 3.1, there exists an invertible $w \in \mathcal{N}$, s.t. $wp = p'$, and $w|_{\mathbf{ker}\, p} = I_{\mathbf{ker}\, p}$.

Let $c = (2\|pu\mathbf{pr}(F)\|)^{-1}$, $u' = p_0 + cpu\mathbf{pr}(F)$, and $v = p_0 + c^{-1}p'$. Then $u = vu'$. By Theorem 1.1, $u'$ is a product of $2\gamma n$ projections. It remains to show that $v$ is a product of $2n$ idempotents.

There exist mutually orthogonal projections $p_1, \ldots, p_n$ s.t. $p_1 + \cdots + p_n = p$, and partial isometries $w_k$, with the property that $p_k = w_k^* w_k$, and $q_k = w_k w_k^* \leqslant q$. Let $p_i' = wp_i$. Note that $pp_i' = pwpp_i = pp'p_i = pp_i = p_i$, hence

$$p_j p_i' = p_j pp_i' = p_j p_i = \begin{cases} p_j, & j = i, \\ 0, & j \neq i. \end{cases}$$

Furthermore,

$$p_j' p_i' = wp_j p_i' = \begin{cases} p_j', & j = i, \\ 0, & j \neq i. \end{cases}$$

Finally, $p_i' p_j = wp_i p_j = 0$ if $i \neq j$, $= p_j'$ otherwise.

Note that $q_k$ is orthogonal to $p_0 + p$ for each $k$, hence, by Lemma 3.2, for $1 \leqslant k \leqslant n$ there exists an idempotent $h_{k1} \in \mathcal{N}$ s.t. $h_{k1}(p_0 + p) = w_k$. Moreover, $\mathbf{pr}(\mathbf{ran}\, p_k')$ is at positive angles with $p_0$, and orthogonal to $\sum_{i \neq k} p_i$, hence, by Lemma 3.2 again, there exists an idempotent $h_{k2} \in \mathcal{N}$ s.t. $h_{k2}(q_k + p_0 + \sum_{i \neq k} p_i) = vw_k^*$. Note that $\mathbf{ran}\, h_{k1} = \mathbf{ran}\, q_k$ and $\mathbf{ran}\, h_{k2} = \mathbf{ran}\, p_k'$ (by definition, $v$ takes $\mathbf{ran}\, p_k$ onto $\mathbf{ran}\, p_k'$). Then $h_{k1}' = h_{k1} + p_0 + \sum_{i \neq k} p_i$ is an idempotent. Furthermore, let $\tilde{p}_k = \mathbf{pr}(\mathbf{ran}\, p_0 p_k')$. By [10], there exists an idempotent $p_k'' \in \mathcal{N}$ s.t. $\tilde{p}_k p_k'' = p_k''$, $p_k'' \tilde{p}_k = \tilde{p}_k$, and

$$p_k''\Big(p_0 - \tilde{p}_k + \sum_{i \neq k} p_i + p_k' + q\Big) = 0.$$

Then $h_{k2}' = h_{k2} + p_0 - \tilde{p}_k + p_k'' + \sum_{i \neq k} p_i$ is an idempotent, and

$$h_{k2}' h_{k1}' = p_k' v p_k + \sum_{i \neq k} p_i + p_0.$$

From the above formulae on the products of $p_i$'s and $p_j'$'s, $\prod_{k=1}^{n} (h_{k2}' h_{k1}') = v$. Thus, $v$ is a product of $2n$ idempotents. $\square$

**Proposition 3.3.** *Any product of idempotents in a von Neumann algebra $\mathcal{N}$ belongs to the norm closure of invertible elements of $\mathcal{N}$.*

**Proof.** Combine Theorem 1.6 with Theorem 1 of [7]. $\square$

**Proposition 3.4.** *Suppose $\mathscr{N}$ is a von Neumann algebra without finite type* I *summands, acting on a Hilbert space $H$. Then for every $c > 0$ there exists $u \in \mathscr{N}$ s.t. $\mathbf{ran}\, u \prec \mathbf{ker}\, u$, and $\|u\| = c$, yet $u$ cannot be written as a product of less than three idempotents in $B(H)$.*

**Lemma 3.5.** *Suppose $\mathscr{N}$ is a von Neumann algebra without finite type* I *summands, and $p_1, \ldots, p_n \in \mathscr{N}$ are mutually orthogonal, mutually equivalent projections, such that $p_1 + \cdots + p_n = \mathbf{1}$. Then $p_1 \mathscr{N} p_1$ has no finite type* I *summands.*

**Proof.** Suppose, for the sake of contradiction, there exists a central projection $z_1 \in p_1 \mathscr{N} p_1$ s.t. $z_1 p_1 \mathscr{N} p_1 = z_1 \mathscr{N}$ is of finite type I. For $2 \leqslant k \leqslant n$, there exists $u_k \in \mathscr{N}$ s.t. $u_k u_k^* = p_1$ and $u_k^* u_k = p_k$. Then $z = z_1 + \sum_{k=2}^n u_k^* z_1 u_k$ is a central projection in $\mathscr{N}$. Indeed, any $x \in \mathscr{N}$ can be written as $x = \sum_{k,\ell=1}^n p_k x p_\ell$. Note that

$$z p_k x p_\ell = u_k^* z u_k u_k^* u_k x u_\ell^* u_\ell = u_k^* (z p_1)(u_k x u_\ell^*) u_\ell = u_k^* (u_k x u_\ell^*) z_1 u_\ell = p_k x p_\ell z,$$

which shows the centrality of $z$. Then $z\mathscr{N}$ is isomorphic to $M_n(z_1 \mathscr{N})$, hence of finite type I. $\square$

**Proof of Proposition 3.4.** By [15], Proposition V.1.35, any von Neumann algebra $\mathscr{M}$ with no finite type I summands contains a projection $p$ s.t. $p \sim p^\perp$. By Lemma 3.5, $p\mathscr{M}p$ has no finite type I summands. Denoting $\mathbf{1}$ by $p_0^{(0)}$, and applying these results, we obtain the existence of projections $(p_k^{(n)})$ ($n \in \mathbb{N}$, $k \in \{0, 1, 2, 3\}$), such that, for each $n \in \mathbb{N}$, the projections $(p_k^{(n)})$ ($k = 0, 1, 2, 3$) are mutually orthogonal, mutually equivalent, and $\sum_{k=0}^3 p_k^{(n)} = p_0^{(n-1)}$. For $n \in \mathbb{N}$ and $k = 2, 3$, there exist partial isometries $u_k^{(n)} \in \mathscr{N}$ s.t. $u_k^{(n)} u_k^{(n)*} = p_{k-1}^{(n)}$, and $u_k^{(n)*} u_k^{(n)} = p_k^{(n)}$. Define $u \in \mathscr{N}$ by setting $u = c \sum_{n=1}^\infty \sum_{k=2}^3 u_k^{(n)}$ As shown in [5], $u$ cannot be a product of less than three idempotents. $\square$

**Theorem 3.6.** *Suppose $\mathscr{N}$ is a separably acting von Neumann factor, and $u \in \mathscr{N}$ is such that both $\mathbf{pr}(\mathbf{ker}\, u)$ and $\mathbf{pr}(\mathbf{ran}\, u)^\perp$ are equivalent to $\mathbf{1}$. Then $u$ is a product of three idempotents.*

In the statement of the theorem, $\mathscr{N}$ is a factor of type $\mathrm{I}_\infty$, $\mathrm{II}_\infty$, or III. The type $\mathrm{I}_\infty$ case was handled in [5].

**Lemma 3.7.** *Suppose $\mathscr{N}$ is an infinite von Neumann factor, and the projections $p_1, p_2 \in \mathscr{N}$ are such that* (1) $\|p_1 p_2\| < 1$, *and* (2) *$p_1^\perp$ and $p_2^\perp$ are infinite. Then there exists an infinite projection $p \in \mathscr{N}$ s.t. $\|pp_i\| < 1$ for $i = 1, 2$.*

**Proof.** If $p_1$ is finite, then $p = p_1^\perp \wedge p_2^\perp$ is an infinite projection. Indeed,

$$p_2^\perp - p_2^\perp \wedge p_1^\perp \sim p_2^\perp \vee p_1^\perp - p_1^\perp \leqslant \mathbf{1} - p_1^\perp = p_1,$$

hence $p_2^\perp \prec p_2^\perp \wedge p_1^\perp + p_1$. If $p$ is a finite projection, then $p_2^\perp$ is also finite, which yields a contradiction.

Thus, we can assume that both $p_1$ and $p_2$ are infinite, and $p_1^\perp \vee p_2^\perp$ is finite. Consider the polar decomposition $p_1 p_2 = UA$, where $A$ is a positive operator, and $U$ is a partial isometry. Suppose first that there exists $c > 0$ s.t. $q = \chi_{[c,1]}(A)$ is an infinite projection. Then $E = \mathbf{ran}\, q$ is a subspace of $F = \mathbf{ran}\, p_2$. As $U$ is a partial isometry, $p_1(E)$ is orthogonal to $p_1(F \ominus E)$. Then $p_1^{\perp}(E)$ is orthogonal to $p_1^{\perp}(F \ominus E)$. Indeed, for $\xi \in E$ and $\eta \in F \ominus E$,

$$\langle p_1^{\perp}\xi, p_1^{\perp}\eta\rangle = \langle \xi, \eta\rangle - \langle p_1\xi, p_1\eta\rangle = 0.$$

Let $p = \mathbf{pr}(p_1^{\perp}(E))$. Then $p$ is infinite (it is equivalent to $q$), $pp_1 = 0$, and $\|pp_2\| = \|p_1^{\perp}q\| \leqslant \sqrt{1-c^2}$ (the last inequality follows from the Pythagorean Theorem). Thus, $p$ has the desired properties.

Suppose $\chi_{[0.1,1]}(A)$ is a finite projection. Then $q = p_2\chi_{[0,0.1]}(A)$ is infinite, and so is $q_1 = p_1 - \mathbf{pr}(\mathbf{ran}(p_1(p_2 - q)))$. Moreover, for any $\xi \in \mathbf{ran}\, q$,

$$\|p_1^{\perp}\xi\|^2 = \|\xi\|^2 - \|p_1\xi\|^2 = \|\xi\|^2 - \|A\xi\|^2 \geqslant 0.99\|\xi\|^2.$$

Therefore, $p_1^{\perp}$ is injective on $\mathbf{ran}\, q$, and $q_2 = \mathbf{pr}(\mathbf{ran}(p_1^{\perp}q))$ is infinite.

Consider a partial isometry $u$ s.t. $u^*u = q_2$ and $uu^* = q_1$. Let $v = (uq_2 + q_2)/\sqrt{2}$ (this is a partial isometry), and $p = \mathbf{pr}(\mathbf{ran}\, v)$. Then $p \sim q_2$, hence it is infinite. For any $\xi \in \mathbf{ran}\, p$, $\|q_1\xi\| = \|q_2\xi\| = \|\xi\|/\sqrt{2}$. Therefore, $\|q_2p\| = \|q_1p\| = 2^{-1/2}$. Note that $p_1 - q_1$ is orthogonal to both $q_1$ and $q_2 \leqslant p_1^{\perp}$, hence $(p_1 - q_1)p = 0$, and $\|p_1p\| = \|q_1p\| = 2^{-1/2}$. Furthermore, $(p_2 - q)p = 0$, and therefore, $p_2p = qp$. Since $\|qp\| \leqslant \|q_2p\| + \|q - q_2\|$, it remains to establish that $\|q - q_2\| < 0.11$.

By Akhiezer-Glazman formula (see e.g. [11,17]),

$$\|q - q_2\| = \max\{\|q_2^{\perp}q\|, \|q^{\perp}q_2\|\}. \tag{3.1}$$

Recall that $q_2$ was defined in such a way that $\|q_2^{\perp}q\| = \|p_1q\| \leqslant 0.1$. Furthermore, for any norm one $\xi \in \mathbf{ran}\, q_2$ there exists $\eta \in \mathbf{ran}\, q$ s.t. $\xi = q_2\eta$, and $\|\eta\|^2 \leqslant 1/(1 - 10^{-2})$. Then

$$\|q^{\perp}\xi\|^2 \leqslant \|\xi - \eta\|^2 = \|\eta\|^2 - \|\xi\|^2 \leqslant 1/99,$$

and therefore, $\|q^{\perp}q_2\| \leqslant 99^{-1/2} < 0.11$. Plugging the estimates for $\|q^{\perp}q_2\|$ and $\|q_2^{\perp}q\|$ into (3.1), we complete the proof. □

**Proof of Theorem 3.6.** Let $p_1 = \mathbf{pr}((\mathbf{ker} u)^{\perp})$ and $p_2 = \mathbf{pr}(\mathbf{ran}\, u)$. By Lemma 3.7, there exists $p_3 \sim \mathbf{1}$ s.t. $\|p_3p_1\|, \|p_3p_2\| < 1$. Denote by $v$ the partial isometry satisfying $vv^* = p_3$, $v^*v = p_1$. By Lemma 3.2, there exist idempotents $q_2, q_3 \in \mathcal{N}$ s.t. $q_2p_1 = v$, and $q_3p_3 = uv^*$. Then $q_3q_2p_1 = up_1 = u$. □

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